Plane Wave Soln of $\vec{E}$ & $\vec{B}$

suppose $\vec{E}$ & $\vec{B}$ are propogate in x, y, z dirn

$$\vec{E} = E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{B} = B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

These soln's must satisfy the maxwell's eqn as they are the solution of Maxwell's eqn.

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \boxed{\vec{K}\cdot\vec{E} = 0} \quad \text{———(9)}$$

$i(K\cdot E) = 0$

∇-operator apply on plane wave. (it operates on space part)

After operation it gives ik.

& time operator gives $-i\omega$.

$$\boxed{\begin{array}{l}\nabla \rightarrow iK \\ \frac{\partial}{\partial t} \rightarrow -i\omega\end{array}}$$

Vibration $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow i(K\cdot B) = 0 \Rightarrow \boxed{\vec{K}\cdot\vec{B} = 0}$ ———(10) form (A) & (B).

∴ $\vec{E}$ & $\vec{B}$ are the $\perp^r$ to the wave propagation.

Wave vector tells the dirn of propogation.

So $\boxed{\vec{E} \text{ \& } \vec{B} \text{ are Transvers in Nature.}}$

Put the value of $\vec{E}$ & $\vec{B}$ in eqn (3) & (4),

$$\text{eq}^n(3) \Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\cancel{i}(\vec{K}\times\vec{E}) = -(-i\omega)\vec{B}$$

$$= +\cancel{i}\omega\vec{B}$$

$$\vec{K}\times\vec{E} = \omega\vec{B}$$

$$\boxed{\vec{B} = \left(\frac{\vec{K}\times\vec{E}}{\omega}\right)} \quad \text{———(11)}$$

As $E \perp^r K$ & $B \perp K$ so

$\vec{E}$, $\vec{B}$ & $\vec{K}$ are Mutually perpendicular.

dirn of $\vec{B} = \vec{K}\times\vec{E}$

$eq^n (4) \Rightarrow \quad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$i(\vec{K} \times \vec{B}) = \mu_0 \epsilon_0 (-i\omega) \vec{E}$$

$$\vec{K} \times \vec{B} = -\mu_0 \epsilon_0 \omega \vec{E}$$

$$\boxed{\vec{E} = \frac{-c^2 (\vec{K} \times \vec{B})}{\omega}} \quad \text{——} (12)$$

When wave vector K & mag. field B are given & find E. field $\vec{E}$ then use above relation directely.

dir^n of $\vec{E} \rightarrow \quad -(\vec{K} \times \vec{B})$

from (12) $\rightarrow \quad \vec{E}, \vec{B}, \vec{K}$ are mutually $\perp^r$.

In free space, Electromag. waves are plane waves.

Let $\quad \vec{E} = E \hat{x}$

$\quad \quad \vec{B} = B \hat{y}$

Hence $\quad \vec{K} = k \hat{z}$

Now, $\quad \boxed{\vec{E} = E_0 e^{i(kz - \omega t)} \hat{x}} \quad \quad (\vec{K} \cdot \vec{r} = kz)$

& $\quad \boxed{\vec{B} = B_0 e^{i(kz - \omega t)} \hat{y}}$

- $E = E_0 e^{i(kz - \omega t)} \hat{x}$ —→ dir^n of E. field

ωt —→ freq. of vibration of field is ω

E_0 ↓ amplitud of wave; kz ↓ dir of propogation is $+z$

- If $E = E_0 e^{i(\omega t - kz)} \hat{z}$

still dir^n of ~~field~~ propogation is $+\hat{z}$.

- Either $(\omega t - kz)$ or $(kz - \omega t)$ then dir^n of prop. is $+z$

i.e. b/w ωt & kz, one plus, one minus then $+z$

- If sign of ωt & kz is same then dir^n of propogation $-\hat{z}$

i.e. $\vec{E} = E_0 e^{i(-\omega t - kz)} \hat{z} \longrightarrow -\hat{z}$

$\quad = E_0 e^{i(\omega t + kz)} \hat{z}. \longrightarrow -\hat{z}$

* Among ωt & kx
  same sign $\longrightarrow -\hat{z}$
  opposite sign $\longrightarrow +\hat{z}$

Energy Density :- (u) Energy per unit volume.

Electric field vector gives the Electric energy density;
& magnetic " " " " magnetic " " u_m

$$u_e = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

E → Total elec. field

$$\boxed{u_e = \frac{\varepsilon_0}{2} E^2}$$

&

$$u_m = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau$$

$$\boxed{u_m = \frac{B^2}{2\mu_0}}$$

We have, $\vec{B} = \frac{\vec{K}\times\vec{E}}{\omega}$ $\vec{K} \to K\hat{z}$, $\vec{E} \to E\hat{x}$ $\{\hat{z}\times\hat{x} = \hat{y}\}$

$$\vec{B} = \frac{KE}{\omega}\hat{y}$$

$\boxed{v = \frac{\omega}{K}}$ = wave velocity (in free space)

so $\vec{B} = \frac{E}{c}\hat{y}$

$$\therefore \quad u_m = \frac{B^2}{2\mu_0} = \frac{E^2}{2c^2\mu_0} = \frac{E^2\mu_0\varepsilon_0}{2\mu_0} = \frac{\varepsilon_0 E^2}{2}$$

$$\boxed{u_m = u_e}$$

Mag. field energy = Electric field energy.
In free space, Mag. field & Ele. field carry equal energy. This energy remain in field, i.e. mag. energy remain in mag. field & E. energy remain in Elec. field.

Total Energy density $\boxed{u = u_e + u_m = \varepsilon_0 E^2}$

Energy Flux or Poynting Vector ($\vec{S}$) :-

Poynting Vector is defined as Energy per unit area per unit time carried by the electromagnetic wave.

OR Power per unit area is called Poynting Vector.

Unit :- $J/m^2\text{-sec}$ or W/m^2 (watt/m^2)

$$\vec{S} = \vec{E} \times \vec{H}$$

$$= \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$$\vec{S} = \frac{EB}{\mu_0}\hat{z}$$

dir^n of $E \rightarrow \hat{x}$
$B \rightarrow \hat{y}$
$(\hat{x} \times \hat{y} \rightarrow \hat{z})$

$$B = \frac{E}{c} \Rightarrow \vec{S} = \frac{E^2}{\mu_0 c}\hat{z}$$

$$\vec{S} = \frac{E^2}{\mu_0 c} \times \frac{c}{c}\hat{z} = \frac{E^2 c}{\mu_0 c^2} = \frac{E^2 \cancel{\mu_0} \epsilon_0 c}{\cancel{\mu_0}}$$

$$\vec{S} = c\, \epsilon_0 E^2 \hat{z}$$

$$\boxed{\vec{S} = cu\hat{z}} \qquad (u = \epsilon_0 E^2)$$

This is Relation b/w Poynting vector & energy density.

$\vec{S} \rightarrow$ tells the dir^n of Energy propagation.
$\vec{K} \rightarrow$ " " " Wave "

Generally, dir^n of $\vec{S}$ & $\vec{K}$ matches but not every time.

In present case, dir^n of $\vec{K} = \hat{z}$
dir^n of $\vec{S} = \hat{z}$

In free space, dir^n of Wave propagation is same as dir^n of energy flow.

Electromag. wave not only carry the energy but also carry the momentum.